\[
\phi_{m-1}(z_0) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} ((z - z_k)^m f(z))
\]

\(\phi\) non zero at \(z_0\).

### Tensors?
- Stress tensor.
- Outer product gives \(u\) tensors
- Tensor (rank): 0 - scalar, 1 - vector, 2 - matrix, \(N\) - \(N\) matrix, \(N\) - tensor

### Einstein Notation
- Dot product: invariance to rotations. \(\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}|| ||\mathbf{b}|| \cos \theta\)
- Einstein notation: sum over repeated indices is implicit: \(\mathbf{A} \cdot \mathbf{B} = A_i B_i\)
- Indices cannot appear more than twice within a term.

### "Free" Indices
- \(|\mathbf{A} \times \mathbf{B}| = ||\mathbf{A}|| ||\mathbf{B}|| \sin \theta\)
- \((\mathbf{A} \times \mathbf{B}) = \epsilon_{ijk} A_i B_j\)
- \(\epsilon_{ijk} = \begin{cases} 0 & (i = j) \cup (j = k) \cup (k = i) \\ 1 & (1, 2, 3) \\ -1 & (3, 2, 1) \end{cases}\)
- \(\epsilon_{ijk} \epsilon_{ilm} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}\)

\[\epsilon_{ijk} \epsilon_{klm} = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl})\]

\[\epsilon_{ijk} \epsilon_{ijm} = 2 \delta_{km}\]

### Other Relations
- \(\tilde{A} \times (\tilde{B} \times \tilde{C}) = \epsilon_{ijk} A_j C_k B_i C_m = \tilde{A} \epsilon_{klm} B_l C_m = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) A_j B_i C_m = B_i A_j C_j - C_i A_j B_j = B_i (\tilde{A} \cdot C) - C (\tilde{A} \cdot B)\)
- \(\{\mathbf{B} (\tilde{A} \cdot \tilde{C})\}_{i} = \{\tilde{B} (\tilde{C} \times \tilde{A})\}_{i} = C_i \epsilon_{ijk} A_j B_k = \{\tilde{C} \cdot (\tilde{A} \times \tilde{B})\}_{i}\)

### Outer Product
- \(\tilde{A} \cdot \tilde{B} = (A_x, A_y, A_z) \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}\)

### Directional Derivative
- \((\mathbf{\nabla} \cdot \tilde{A})_{ij} = \frac{\partial}{\partial x_j} A_k\)
- \((\mathbf{\nabla} \times \tilde{A})_{ij} = \epsilon_{ijk} \frac{\partial}{\partial x_j} A_k\)
- \((\mathbf{\nabla} \cdot \tilde{A}) = \tilde{c} \cdot (\mathbf{\nabla} \times \tilde{A})\)

### Derivatives
- \(\phi(\tilde{r})\) scalar function of \(\tilde{r}\)
- \(\tilde{A}(\tilde{R}) = (A_x(x, y, z), A_y(x, y, z), A_z(x, y, z))\) (vector function of \(\tilde{r}\)
meaningful derivatives - must have a meaning that doesn’t depend on x,y,z.
these derivatives always has to do with vector differential operator "nabla" or "del"
Gradient (a vector)
∇φ = rate of changes of φ(r) points in the direction of maximum change
particles tends to goes to higher intensity via the gradient (when laser shine on the particle)

Divergence - local density of net flux
A(\vec{r})
Net flux = \int_S A(\vec{r}) \cdot d\vec{\sigma}
Div[A(\vec{r})]d\tau = \int_S A \cdot d\vec{\sigma}
Div{A} = \nabla \cdot A(\vec{r})

Curl - local density of circulation
curl{A} = \nabla \times A

second derivatives

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\nabla[\phi q] = q \nabla \phi + \phi \nabla q
scalar times vector:
\{ \nabla \cdot (\phi \vec{A}) \}_i = \frac{\partial}{\partial x_j} (\phi A_i) = \phi \frac{\partial A_i}{\partial x_j} + A_i \frac{\partial}{\partial x_i} \phi \Rightarrow \nabla \cdot (\phi \vec{A}) = \phi \nabla \cdot \vec{A} + \vec{A} \cdot \nabla \phi

\{ \nabla \times (\phi \vec{A}) \}_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} (\phi A_i) = \epsilon_{ijk} \phi \frac{\partial A_i}{\partial x_j} + \epsilon_{ijk} \frac{\partial}{\partial x_j} A_k \Rightarrow \nabla \times (\phi \vec{A}) = \phi \nabla \times \vec{A} + (\nabla \phi) \times \vec{A}

vector times vector:
\{ \nabla(\vec{A} \cdot \vec{B}) \}_i = \frac{\partial}{\partial x_j} (A_i B_j) = A_j \frac{\partial}{\partial x_i} B_j + B_j \frac{\partial}{\partial x_i} A_j \Rightarrow \nabla(\vec{A} \cdot \vec{B}) = (\nabla \otimes \vec{B}) \cdot \vec{A} + (\nabla \otimes \vec{A}) \cdot \vec{B}

\{ \nabla \cdot (\vec{A} \times \vec{B}) \}_i = \frac{\partial}{\partial x_j} (\epsilon_{ijk} A_k B_j) = \epsilon_{ijk} B_k \frac{\partial}{\partial x_i} A_j + \epsilon_{ijk} A_j \frac{\partial}{\partial x_i} B_k = B \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})

\{ \nabla \times (\vec{A} \times \vec{B}) \}_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{klm} A_l B_m = \epsilon_{klm} \epsilon_{ijk} \left( B_m \frac{\partial}{\partial x_j} A_l + A_l \frac{\partial}{\partial x_j} B_m \right) = B_j \frac{\partial}{\partial x_j} A_i + A_i \frac{\partial}{\partial x_j} B_j - B_i \frac{\partial}{\partial x_j} A_j - A_j \frac{\partial}{\partial x_j} B_i \Rightarrow \nabla \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \nabla)\vec{A} + \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) - (\vec{A} \cdot \nabla)\vec{B}
Fundamental Theorem of calculus

(1D) \[ \int_a^b f'(x) \, dx = f(b) - f(a) \]
Def Deriv. \[ f'(x) \Delta x = f(x + \Delta x) - f(x) \]
Def Int. \[ \int_a^b F(x) \, dx = \lim_{\Delta x \to 0} \sum_{n=0}^{N-1} F(x_n) \Delta x \quad (x_n = a + n \Delta x) \]

(3D)
\[ \nabla \phi \cdot \Delta \vec{r} = \phi(\vec{r}) + \Delta \phi(\vec{r}) - \phi(\vec{r}) \]
Integral along path (line integral): \[ \int_{\Gamma} F(\vec{r}) \cdot d\vec{\lambda} = \lim_{\Delta \lambda_n \to 0} \sum_{n=1}^{N-1} F(\vec{r}_n) \cdot \Delta \lambda_n \]

If \( \vec{F}(\vec{r}) = \nabla \phi(\vec{r}) \)
\[ \int_{\Gamma} \nabla \phi \cdot d\vec{\lambda} = \lim_{\Delta \lambda_n \to 0} \sum_{n=0}^{N-1} \nabla \phi(\vec{r}_n) \cdot \Delta \lambda_n = \phi(\vec{r}_b) - \phi(\vec{r}_a) \]
Def. of Div. \[ \nabla \cdot \nabla \phi = \int_{S \text{ of } \Delta \tau} A \cdot d\vec{\sigma} \]
\[ \int_{\Gamma} f(\vec{r}) d\tau = \lim_{\Delta \tau_n \to 0} \sum f(\vec{r}_n) \Delta \tau_n \]
if \( f(\vec{r}) = \nabla \cdot \vec{A}(\vec{r}) \)
\[ \int_{\Gamma} \nabla \cdot \vec{A} d\tau = \int_{S \text{ of } \Delta \tau} \vec{A} \cdot d\vec{\sigma} \]
Def: \( \nabla \times \vec{A} \cdot \Delta \vec{\sigma} = \int_{C \text{ of } \Delta \vec{\sigma}} \vec{A} \cdot d\vec{\lambda} \)
\[ \int_{\Gamma} (\nabla \times \vec{A}) \cdot d\vec{\lambda} = \lim_{\Delta \sigma_n \to 0} \sum_n (\nabla \times \vec{A}(\vec{r}_n) \cdot \Delta \vec{\sigma}_n) = \int_{C \text{ of } \Delta \vec{\sigma}} \vec{A} \cdot d\vec{\lambda} \]
\[ \int_{\Gamma} \nabla \phi \cdot d\vec{\lambda} = \phi(\vec{r}_b) - \phi(\vec{r}_a) \]
F. T. of Dirs. (Gauss’ Theorem) \[ \int_{\Gamma} \nabla \cdot \vec{A} d\tau = \int_{S \text{ of } \Delta \tau} \vec{A} \cdot d\vec{\sigma} \]
F.T. of Curls (Stokes’ Theorem) \[ \int_{S} (\nabla \times \vec{A}) \cdot d\vec{\sigma} = \int_{C \text{ of } \Delta \vec{\sigma}} \vec{A} \cdot d\vec{\lambda} \]

functions of complex variables

complex functions of complex variables
\[ f(z) = u(x, y) + iv(x, y) \]

derivative
\[ dz = dx + idy \]
\[ f'(z) dz = Re\{f'\} dx - Im\{f'\} dy + i(Im\{f'\} dx + Re\{f'\} dy) \]
RHS \[ f(z + dz) - f(z) = u(x + dx, y + dy) - u(x, y) + iv(x + dx, y + dy) - v(x, y) \]
LHS \[ \text{RHS} \quad \text{RHS} \quad \text{RHS} \]
\[ u_x = v_y, u_y = -v_x \quad \text{Cauchy-Riemann conditions.} \]

existence of \( f'(z_0) \) requires \( u_x = v_y, u_y = -v_x \) at point \( z_0 \).
Write \[ u_r = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}, u_\theta = \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} \quad \text{and for } v \text{ too} \]
its called Cauchy-Riemann equations, but equality doesn’t imply existence of \( f'(z_0) \).
first order partial derivative of \( u, v \) with respect to \( x, y \ (r, \theta) \) exist everywhere in the neighborhood, and are
continuous at \((x_0, y_0)\) and satisfy Cauchy-Riemann equations

\[ u_x = v_y, u_y = -v_x \quad ru_r = v_y, u_\theta = -rv_r \]

at that point.

Then \(f'(z_0)\) exist, its value being \(f'(z_0) = u_x + iv_x = e^{-i\theta}(u_r + iv_r)\)

analytic functions.

analytic in an open set \(S\) if derivative everywhere in that set.

analytic at a point \(z_0\) if it is analytic in some neighborhood of \(z_0\).

entire function(holomorphic) - function that is analytic at each point in the entire plane.

analytic in domain \(D\) requires continuity, first order partial derivative of \(u, v\) with respect to \(x, y (r, \theta)\) exist everywhere in the neighborhood, and are continuous at \((x_0, y_0)\) and satisfy Cauchy-Riemann equations.

if two functions are analytic in a domain \(D\), their sum and product are both analytic in \(D\), quotient too (when denominator \(\neq 0\)).

composition of 2 analytic functions is analytic

singular point \(z_0\) - fails to be analytic at that point, but is analytic at some point in every neighborhood of \(z \neq z_0\)

an analytic function from a well behaved function of a real variable \(x\) by replacing \(x\) with \(z\)

integral

\[ \oint_C \frac{f(z)}{z - z_0} dz = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z} dz. \]

Make \(C\) a circle around \(z_0\) as small as possible. \(\oint_C \frac{f(z)}{z - z_0} dz = \oint_{C_r} \frac{f(z)}{z} dz = 2\pi i f(z)\)

function \(f\) and its conjugate are both analytic \(\rightarrow f(z) = c_0 + ic_1\).

Harmonic: in given domain, has continuous partial derivatives of the 1st, 2nd order and satisfy Laplace’s eq.

Second way to prove harmonic: \(f(z) = u(x, y) + iv(x, y)\) is analytic in a domain \(D\), then \(u, v\) are harmonic in \(D\).

Laplace’s eq. \(u_{xx} + u_{yy} = 0\), function \(v\) too.

\(r^2 u_{rr} + ru_r + u_{\theta\theta} = 0\), function \(v\) too.

Reflection principle

\(f(\overline{z}) = \overline{f(z)}\), On real axis \(f(z) = f(\overline{z})\)

\(d\)

periodic by \(2\pi i\).

\(e^z\) can be negative.

output of \(e^z\) can be any non-zero complex number.

\(\log z = \text{Log} z + 2n\pi i\)

Notice, the function is log, not ln! (don’t make sense but it’s defined this way)

\[ \frac{d}{dz} \log z = \frac{1}{z} \quad z \neq 0, \alpha < \arg z < \alpha + 2\pi \]
For Log $z$, Arg $z \in \{-\pi, \pi\}$, domain slightly diff.

\[
\log(z_1 z_2) = \log z_1 + \log z_2
\]

\[
\arg(z_1 z_2) = \arg z_1 + \arg z_2
\]

$z \neq 0$, $c$ any complex number. power function

\[
z^c = e^{c \log z}, \quad z \neq 0.
\]
can be multiple-valued.

\[
\frac{d}{dz} z^c = cz^{c-1}
\]

\[
\frac{d}{dz} e^z = e^z \text{ log } c, \quad \text{or } \frac{d}{dc} z^c = z^c \log z.
\]

Zeros of sin, cos are the zeros of real line.

\[
\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}
\]

\[
\sinh y = \frac{e^y - e^{-y}}{2}, \quad \cosh y = \frac{e^y + e^{-y}}{2}
\]

\[
\cosh^2 y - \sinh^2 y = 1
\]

\[
\sin z = \sin x \cosh y + i \cos x \sinh y
\]

\[
\cos z = \cos x \cosh y - i \sin x \sinh y
\]

where $z = x + iy$

\[
\sin z, \cos z \text{ not bounded on complex plane.}
\]

\[
|\sin z|^2 = \sin^2 x + \sinh^2 y
\]

\[
|\cos z|^2 = \cos^2 x + \sinh^2 y
\]

\[
\frac{d}{dz} \sinh z = \cosh z, \quad \frac{d}{dz} \cosh z = \sinh z
\]

\[-i \sinh(iz) = \sin z, \quad \cosh(iz) = \cos z
\]

\[-i \sin(iz) = \sinh z, \quad \cos(iz) = \cosh z
\]

zeros of sinh at $i(n\pi)$, zeros of cosh at $i(n\pi + \frac{\pi}{2})$.

\[-i \sinh(iz) = \sin z, \quad \cosh(iz) = \cos z
\]

hyperbolic functions periodic with period $2\pi i$.

\[
\sinh(-z) = -\sinh z, \quad \cosh(-z) = \cosh z
\]

\[
\cosh^2 z - \sinh^2 z = 1
\]

\[
\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2
\]

\[
\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2
\]

\[
\sinh = \sinh x \cos y + i \cos x \sin y
\]

\[
\cosh = \cosh x \cos y + i \sin x \sin y
\]

\[
|\sinh z|^2 = \sinh^2 x + \sin^2 y
\]

\[
|\cosh z|^2 = \cosh^2 x + \cos^2 y
\]

\[
\sinh z = 0 \iff z = n\pi i
\]

\[
\cosh z = 00 \iff z = \left(\frac{\pi}{2} + n\pi\right) i
\]

\[
\frac{d}{dz} \tanh z = \sech^2 z, \quad \frac{d}{dz} \coth z = -\csch^2 z
\]

\[
\frac{d}{dz} \sech z = -\sech z \tanh z, \quad \frac{d}{dz} \csch z = -\csch z \coth z
\]

inverse tri, hyperbolic

\[
\sin^{-1} z = -i \log[iz + \sqrt{1 - z^2}]
\]

\[
\cos^{-1} z = -i \log[z + i\sqrt{1 - z^2}]
\]

\[
\tan^{-1} z = \pm \frac{i \log(1 + iz)}{2} \pm \frac{i}{1 - z^2}
\]

they are multiple-valued

\[
\frac{d}{dz} \sin^{-1} z = \frac{1}{\sqrt{1 - z^2}}
\]
\[
\frac{d}{dz} \cos^{-1} z = \frac{-1}{\sqrt{1 - z^2}} \\
\frac{d}{dz} \tan^{-1} z = \frac{1}{1 + z^2} \\
sinh^{-1} z = \log[z + \sqrt{z^2 + 1}] \\
cosh^{-1} z = \log[z + \sqrt{z^2 - 1}] \\
tanh^{-1} z = \frac{1}{2} \log \frac{1 + z}{1 - z}
\]

\[w'(t) = u'(t) + iv'(t)\quad (1)\]

\[
\frac{d}{dt} w(t)^2 = 2w(t)w'(t)\quad \text{(by 1)}
\]

\[
\frac{d}{dt} e^{z_0 t} = z_0 e^{z_0 t}\quad \text{(by 1)}
\]

* mean value theorem no longer apply.

\[
\int_a^b w(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt
\]

existence of integral ensured if functions are piecewise continuous.

mean value theorem for integral doen't apply either.

Arc - is simple if it does not cross itself.

simple closed curve, Jordan curve. Positively oriented if in counterclockwise direction.

contour - piecewise smooth arc.

line integral

\[
\int_C f(z) dz = \int_a^b f[z(t)] z'(t) dt
\]

\[
\int_C z_0 f(z) dz = z_0 \int_C f(z) dz
\]

\[
\left| \int_a^b w(t) dt \right| \leq \int_a^b |w(t)| dt
\]

\[|f(z)| \leq M \Rightarrow \left| \int_C f(z) dz \right| \leq ML\]

Cauchy-Goursat theorem

if \( f \) analytic at all points interior 2 and on a simple closed contour \( C \), then \( \int_C f(z) dz = 0 \). converse also true.

analytic throughout a simply connected domain, then true for every closed contour \( C \) lying in \( D \). \( f \) has antiderivative everywhere in \( D \).

analytic on and within \( C \), \( z_0 \) any point interior to \( C \).

\[f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{z - z_0}\]

\[\int_C \frac{f(z) dz}{z - z_0} = 2\pi i f(z_0)\]

\[f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z) dz}{(z - z_0)^{n+1}}\quad n \in \mathbb{N}_0\]

\[f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(s) ds}{(s - z)^{n+1}}\quad n \in \mathbb{N}_0\]

\( f \) analytic at \( z_0 \) \( \Rightarrow \) its derivatives of all orders are analytic at \( z_0 \) too

\[|f^{(n)}(z_0)| \leq \frac{n!M_R}{R^n}\quad n \in \mathbb{N} \text{ where } M_R = \text{maximum value of } |f(z)| \text{ on } C_R\]

6
Liouville’s theorem
If a function \( f \) is entire and bounded in complex place, then \( f(z) \) is constant throughout the plane.

Fundamental theorem of algebra
Any polynomial with degree \( n \leq 1 \) has at least 1 zero.

If \( f \) is analytic and not constant in a given domain \( D \), then \( |f(z)| \) has no maximum value in \( D \).

Taylor series (complex)
\[
f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{(over } n!)\]

negative power of \( (z - z_0) \)
e^z, compose with \( (z + n) \), multiply by \( e^n \) and taylor expand.

Laurent’s theorem
If \( f \) is analytic on annular domain, \( C (+) \) closed contour in that domain. Then at each point on the domain,
\[
f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n\quad c_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}}
\]

Uniform convergence of Taylor series - all points on the largest circle centered at \( z_0 \).

Multiplication of power series
\[
[f(z)g(z)]^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z)g^{(n-k)}(z) \quad n \in \mathbb{N}
\]

\[
c_n = \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} \cdot \frac{g^{(n-k)}(z_0)}{(n-k)!} = \sum_{k=0}^{n} a_k b_{n-k}
\]

Residue
A singular point \( z_0 \) is isolated if there is a deleted \( \epsilon \) neighborhood of \( z_0 \) throughout which \( f \) is analytic.
\[
\int_C f(z)dz = 2\pi i b_1 = 2\pi i \text{Res}_{z=z_0} f(z)
\]

\( \text{Res}_{z=z_0} f(z) \) is power series in functions of \( z - z_0 \)
Partial fraction can help
Point at infinity is said to be an isolated singular point of \( f \).
\[
\text{Res}_{z=\infty} f(z) = -\text{Res}_{z=0} \left( \frac{1}{z} f \left( \frac{1}{z} \right) \right)
\]

Removable (all negative power of \( z - z_0 \) are 0), essential (infinitely many nonzero negative power of \( z - z_0 \)), or a pole (finite nonzero negative power of \( z - z_0 \), largest negative power \( m \) means pole of order \( m \)).

Picard’s theorem - each neighborhood of an essential singular point, a function assumes every finite value,
with one possible exception, an infinite number of times.

residue at poles

\( z_0 \) is a pole of order \( m \) of \( f \), equivalent to

\( f(z) \) can be written in the form

\[
f(z) = \frac{\phi(z)}{(z-z_0)^m} \quad m \in \mathbb{N}
\]

where \( \phi(z) \) is analytic and nonzero at \( z_0 \).

\[
\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}
\]

zero of order \( m \) at \( z_0 \): \( f(z_0) = \ldots = f^{(m-1)}(z_0) = 0, f^{(m)}(z_0) \neq 0 \)

if \( f(z) = \frac{p(z)}{q(z)} \) where singularity at \( z_0 \), if \( p(z_0) \neq 0, q(z_0) = 0, q'(z_0) \neq 0 \), then

\[
\text{Res}_{z=z_0} f = \frac{p(z_0)}{q'(z_0)}
\]

if \( f \) analytic at \( z_0 \), \( f(z_0) = 0 \) but \( f(z) \) is not identically equal to zero in any neighborhood of \( z_0 \).

Then \( f(z) \neq 0 \) throughout some deleted neighborhood of \( z_0 \).

if \( f \) analytic throughout neighborhood \( N_0 \) of \( z_0 \), \( f(z_0) = 0 \) at every point in domain \( D \) or line segment \( L \) containing \( z_0 \), then \( f(z) \equiv 0 \) in \( N_0 \)

Then \( f(z) \neq 0 \) throughout some deleted neighborhood of \( z_0 \).

if \( z_0 \) a removable singular point of \( f \), then \( f \) is bounded and analytic in some deleted neighborhood of \( z_0 \).

\( f \) bounded, analytic in deleted \( \epsilon \) neigh of \( z_0 \). If \( f \) is not analytic at \( z_0 \), then it has a removable singularity there.

essential singular point: in each deleted neigh of the ESP, \( f \) assumes values arbitrarily close to any given number.
if \( z_0 \) is a pole, then \( \lim_{z \to z_0} f(z) = \infty \).

\[
\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z)
\]

\( C_R \) semi-circular arc, \( z_k \) are singular points inside the semi circle.

Cauchy principal value (P.V.)

\[
P.V. \int_{-\infty}^{\infty} f(x)dx = \lim_{R \to \infty} f(x)dx
\]
The improper integral doesn't always converge even if its P.V. exists.

For a simple pole on x-axis, \( \lim_{\rho \to 0} \int_{C_\rho} f(z)dz = -B_0 \pi i \)
if all singularity on one side of the x-axis, we can make a contour with two concentric circles connected by a line along the x-axis.

definite integral with sin,cos

\[
\int_{0}^{2\pi} F(\sin \theta, \cos \theta)d\theta = \int_{C} F \left( \frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2i} \right) \frac{dz}{iz}
\]

argument principle
meromorphic in domain \( D \) if analytic throughout except poles.
if \( f \) meromorphic in \( D \) interior to \( C \), analytic and nonzero on \( C \), then winding number

\[
\Gamma = \frac{1}{2\pi} \Delta C \arg f(z) = Z - P
\]

where \( Z \) is the number of zero, \( P \) number of poles of \( f \) inside \( C \), counting multiplicities.
branch point: points that cause that function to be multi-valued. \( f(z) \) can be finite/infinity at branch points. At branch points, if we close an arbitrarily small loop around that point, it travels and end up at a different value.

branch cut: \( \sqrt{z} \) has 2 Riemann sheets. The whole thing is a Riemann surface.

if taylor expand \( \sqrt{z} \) at \( z_0 > 0 \), uniform convergence in the circle of radius \( z_0 \) centered at \( z_0 \)

Product of fractional powers.

\[ f(z) = \sqrt{z^2 - 1} = (z - 1)^{\frac{1}{2}} (z + 1)^{\frac{1}{2}} \]

branch cut: \((1, \infty)\) or \((-\infty, -1) \cup (1, \infty)\)

In general \( f(z) = (z - z_0)^{\frac{m}{n}} \) where the fraction is irreducible. function has \( m \) riemann sheets

\[ f(z) = (z - z_0)^{\alpha}(z - z_2)^{\beta}(z - z_3)^{\gamma} \]

\# riemann sheets is lowest common denominator of \( \alpha, \beta, \gamma \)

if \( \alpha + \beta + \gamma = \mathbb{Z} \) - can join the branch points with finite branch cut.

sums: \( f(z) = \sqrt{z - 1} + \sqrt{z + 1} \) has 4 Riemann sheets.

\# of Riemann sheets is the product of the number of Riemann sheets for each. term

Poles: points where function diverge but doesn’t cause multivalue.

\( f(z) \) has finitely many negative powers of \( (z - z_0)^m \).

Essential singularity: infinitely many negative powers of \( (z - z_0)^m \)

Laurent Series

Starting from CIF \( f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')}{z' - z} dz' \)

if all singularities inside \( C_r \) are not branch points or if they include branch pts, but these can be joined by a finite branch cuts. Then the contour integral is just the two circle line integrals (closed)).

(And opposite direction)

\[
\frac{1}{2\pi i} \oint_{C_R} \frac{f(z')}{z' - z} dz' - \frac{1}{2\pi i} \oint_{C_r} \frac{f(z')}{z' - z} dz'
= \frac{1}{2\pi i} \oint_{C_R} \frac{f(z')}{(z' - z_0) - (z - z_0)} dz'
- \frac{1}{2\pi i} \oint_{C_r} \frac{f(z')}{(z' - z_0) - (z - z_0)} dz'
= \frac{1}{2\pi i} \oint_{C_R} \frac{f(z')}{z' - z_0 - \frac{z - z_0}{z' - z_0}} dz'
+ \frac{1}{2\pi i} \oint_{C_r} \frac{f(z')}{z' - z_0 - \frac{z - z_0}{z' - z_0}} dz'
= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{C_R} \frac{f(z')}{(z' - z_0)^{n+1}} dz' (z - z_0)^n + \sum_{n=0}^{\infty} \frac{1}{2\pi i} \oint_{C_r} \frac{f(z')}{(z' - z_0)^{n+1}} dz' (z - z_0)^{n-1}
= \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n= -\infty}^{1} a_n (z - z_0)^n
= \sum_{n= -\infty}^{\infty} a_n (z - z_0)^n
a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_0)^{n+1}} dz'
\]
we can find a LS around a pole.
Taylor series on the region of circle centered at origin with radius reaching first singularity.
Laurent series on other regions (but if there’s branch points, laurent series may not be valid).
remember technique: $|z - 2| > 1, f(z) = \frac{1}{z-2} + \frac{1}{z^2-2}$
remember technique: when $z_0$ is in middle between singularity, partial fraction is not needed.
$f(z) = \sqrt{z^3 - 4}$ expand at $z_0 = 0$
$|z| < 2, f(z) = \pm i \sqrt{1 - \frac{z^2}{4}}$
Use binomial expansion $(1 + x)^{\frac{1}{2}} = ..$

\[ \int_C f(z)dz = 2\pi i b_1 = 2\pi i \text{Res}_{z=z_0} f(z) \]
\[ \text{Res}_{z=\infty} f(z) = -\text{Res}_{z=0} \left( \frac{1}{z^2} f \left( \frac{1}{z} \right) \right) \]
If all singularities inside $C$ are poles (not branch points), then the "bridge" subcontours cancel.
\[ a_{-1} = \text{Res}_{z=z_k} f(z) = \lim_{z \to z_k} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} ((z - z_k)^m f(z)) \]
\[ \text{Res}_{z=z_k} f(z) = \lim_{z \to z_k} (z - z_0) f(z) \]
3 types of integrals
\[ I_1 = \int_{-\infty}^{\infty} f(x)dx \]
Do this type of integral with semi circle with $R \to \infty$ contour lying on x-axis.
\[ \oint_{C_R} f(z)dz = I_1 + i \int_{0}^{\pi} R \to \infty \lim_{r \to 0} z f(z)d\theta \]
if $f(z) \to 0$ faster than $\frac{1}{z}$ as $z \to \infty$
then $I_1 = \oint_{C_R} f(z)dz = 2\pi i \sum_{k \text{ above x-axis}} \text{Res} \{ f(z), z_k \} = -2\pi i \sum_{k \text{ below x-axis}} \text{Res} \{ f(z), z_k \}$

\[ \int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2} = 2\pi i \lim_{z \to ia} \frac{d}{dz} \left( \frac{(z - ia)^2}{(z^2 + a^2)^2} \right) = -4\pi i (z + ia)^{-3} |_{ia} = \frac{\pi}{2a^3} \]
\[ I_2 = \int_{-\infty}^{\infty} f(x)e^{ikx}dx \quad k \in \mathbb{R} \]
for $e^{ikx} = e^{ikx - ky}$ For $k > 0$, use semicircle above x-axis.
For $k < 0$, use semicircle below x-axis.

\[ \text{example: } k > 0, \oint_{C_R} f(z)e^{ikz}dz = I_2 - \lim_{R \to \infty} \int_{0}^{2\pi} *d\theta \]
Second term $\to 0$ if $M(R) \to 0$ as $R \to \infty$. ($f(z) \to 0$ as $|z| \to \infty$)
\[ k > 0 : \quad 2\pi i \text{Res} \left\{ f(z)e^{ikz}, \forall z_0 \right\} = I \quad \text{(upper semi-circle)} \]
\[ k < 0 : \quad -2\pi i \text{Res} \left\{ f(z)e^{ikz}, \forall z_0 \right\} = I \quad \text{(lower semi-circle, negative sign cuz contour counterclockwise)} \]
\[ I = \int_{0}^{\infty} \frac{\sqrt{x}}{x^2 + a^2} dx \quad a > 0 \]
Contour integral:
\[ \lim_{\epsilon \to 0, R \to \infty} \int_{\epsilon}^{R} \frac{r^{\frac{1}{2}}dr}{r^2 + a^2} + \lim_{R \to \infty} \int_{0}^{2\pi} \frac{R^{\frac{1}{2}}e^{i\theta}}{R^2e^{2i\theta} + a^2} + \lim_{\epsilon \to 0, R \to \infty} \int_{R}^{\epsilon} \frac{r^{\frac{1}{2}}e^{i\theta}}{r^2e^{i4\pi} + a^2} d\theta \]
\[ \lim_{\epsilon \to 0} \int_{2\pi}^{0} \frac{e^{\frac{1}{2}e^{i\theta}} + a^2 e^{i\theta}}{e^{2i4\pi} + a^2} d\theta \]
\[ 2I = \frac{1}{2} \int_C \frac{z^2}{z^2 + a^2} \, dz = \pi i \text{Res} \left\{ \frac{z^2}{z^2 + a^2}, ae^{i\theta}, ae^{i2\pi} \right\} = \pi i \sum_{n=0}^{\infty} z_{2n} \frac{z^2}{a^2}, ae^{i\theta} = \pi i \left( \frac{e^{-i\frac{z}{2}}}{2\sqrt{a}} + \frac{e^{i\frac{z}{2}}}{2\sqrt{a}} \right) = \frac{\pi i}{\sqrt{2a}} \]

\[
\int_0^{\pi} x^\alpha \frac{x}{(x^n + a^n)^M} \, dx
\]

if \( n = m \)

or if we just have \( \int_0^{\pi} x^\alpha \frac{x}{(x^n + a^n)^N} \, dx \)

pizza slice containing first singularity.

example

\[ I = \int_0^{\pi} \frac{\sqrt{x}}{x^2 + a^2} \, dx \]

contour integral, upper semi circle, with bump at the origin.

\[
\lim_{\epsilon \to 0, R \to \infty} \int_\epsilon^R \frac{r^2 e^{i\theta}}{r^2 + a^2} \, dr = 0 + \lim_{\epsilon \to 0, R \to \infty} \int_R^{\infty} \frac{r^2 e^{i\theta}}{r^2 + a^2} \, dr
\]

= \( (1 + e^{i\frac{\pi}{4}})I \) = \( 2\pi i \text{Res} \left\{ \frac{z^2}{z^2 + a^2}, ae^{i\frac{\pi}{4}} \right\} = \pi i z^{-\frac{1}{2}} |_{ae^{i\frac{\pi}{4}}} = \frac{\pi i}{\sqrt{a}} e^{-\frac{\pi}{4}} = \frac{\pi}{\sqrt{a}} e^{i\frac{\pi}{4}} \)

LHS is \( 2 \cos \left( \frac{\pi}{4} \right) e^{i\frac{\pi}{4}} I \) \( \Rightarrow I = \frac{\pi}{\sqrt{2a}} \)

\[
\int_C \ln zf(z) \, dz = \lim_{\epsilon \to 0, R \to \infty} \int_\epsilon^R \ln rf(r) \, dr + \lim_{\epsilon \to 0, R \to \infty} \int_0^{2\pi} \ln(Re^{i\theta}) f(Re^{i\theta}) iRe^{i\theta} \, d\theta + \lim_{\epsilon \to 0, R \to \infty} \int_{\epsilon}^R \ln(e^{i2\pi}) f(e^{i2\pi}) iRe^{i2\pi} +
\]

\[
= \lim_{\epsilon \to 0} \int_0^{2\pi} \ln(e^{i\theta}) f(e^{i\theta}) i e^{i\theta} \, d\theta
\]

2nd term goes \( \to 0 \) if \( f(z) \cdot \frac{1}{z^\delta} \) with \( \delta > 0 \) as \( |z| \to \infty \) \( (f(z) \to 0 \) faster than \( \frac{1}{z^\delta} \) as \( |z| \to \infty \))

4th term goes \( \to 0 \) if \( f(0) \) is well defined.

LHS becomes \(-2\pi i \sum_n \text{Res} \left\{ \ln zf(z), z_n \right\} \, \theta \in (0, 2\pi) \)

example

\[ I = \int_0^{\pi} \frac{dx}{x^3 + a^3} \quad a > 0, \text{integral real, positive, } \alpha a^{-2} \]

contour integral of \( \ln zf(z) \) with connected CC \( C_R \), clockwise \( C_\epsilon \)

\[
\lim_{\epsilon \to 0, R \to \infty} \int_\epsilon^R \frac{\ln r}{r^3 + a^3} \, dr + \lim_{\epsilon \to 0, R \to \infty} \int_R^{\infty} \frac{\ln r - i2\pi}{r^3 e^{i2\pi} + a^3} \, dr e^{i2\pi}
\]

\[-2\pi i \Rightarrow I = -\sum_{n=0}^{\infty} \text{Res} \left\{ \frac{\ln z}{3a^2}, ae^{i\frac{\pi}{3}}, ae^{i2\frac{\pi}{3}}, ae^{i3\frac{\pi}{3}} \right\} = -\sum_{n=0}^{\infty} \ln \left| \frac{3a^2}{3a^2 e^{i\frac{\pi}{3}}} + \frac{3a^2 e^{i2\frac{\pi}{3}}}{3a^2 e^{i3\frac{\pi}{3}}} \right| = 2\pi
\]

\[
= \frac{\ln a + i\frac{\pi}{3}}{3a^2 e^{i\frac{\pi}{3}}} + \frac{\ln a + i\frac{\pi}{3}}{3a^2 e^{i\frac{2\pi}{3}}} + \frac{\ln a + i\frac{\pi}{3}}{3a^2 e^{i\frac{3\pi}{3}}} = \frac{2\pi}{3\sqrt{3}a^2}
\]

definite integral with \( \sin, C \) is unit circle

\[
\int_0^{2\pi} F(\sin \theta, \cos \theta) \, d\theta = \int_C F \left( \frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2} \right) \, dz
\]

\[
I = \int_{-\infty}^{\infty} \sin(kx) \, dx = \frac{1}{2i} \left( \int_{-\infty}^{\infty} e^{ikx} \, dx - \int_{-\infty}^{\infty} e^{-ikx} \, dx \right)
\]

\[ k > 0 \quad I = \frac{1}{2i} \left( 0 + 2\pi i \text{Res} \left\{ \frac{e^{-ikx}}{z}, \frac{e^{ikx}}{z} \right\} \right) = \pi
\]

\[ k < 0 \quad I = \frac{1}{2i} \left( -2\pi i \text{Res} \left\{ \frac{e^{ikx}}{z}, \frac{e^{-ikx}}{z} \right\} \right) = -\pi
\]

\[ k = 0 \Rightarrow I = 0, I = \pi \text{sgn}(k) \]
\[
\int_{-a}^{b} \frac{1}{x} \, dx = \lim_{\epsilon_1 \to 0} \int_{-a}^{-\epsilon_1} \frac{1}{x} \, dx + \lim_{\epsilon_2 \to 0} \int_{\epsilon_2}^{b} \frac{1}{x} \, dx = \lim_{\epsilon_1, \epsilon_2 \to 0} \ln \frac{b \epsilon_1}{a \epsilon_2} = \text{anything real depending on how } \epsilon_1, \epsilon_2 \to 0 \nabla \text{ integral not well defined (simple pole)}
\]

We can define the "Principal value" integral \( \mathcal{P} \int_{-a}^{b} \frac{dx}{x} = \lim_{\epsilon \to 0} \left( \int_{-a}^{-\epsilon} \frac{dx}{x} + \int_{\epsilon}^{b} \frac{dx}{x} \right) = \ln \frac{b}{a} \)

1 simple pole on contour: \( \mathcal{P} \oint_{C} f(z) \, dz = \text{difference of 2 contour integral (leap over pole by semi circle).} z = z_0 + \epsilon e^{i \theta} \)

Laurent series for the semi circle integral.
\[
\int_{C_{s}} f(z) \, dz = \lim_{\epsilon \to 0} \int_{\theta_0 + \pi}^{\theta_0} \frac{a-1}{e^{i \theta}} + a_0 + a_1 e^{i \theta} + \ldots \right) i \epsilon e^{i \theta} \, d\theta
\]
\[
= \int_{\theta_0 + \pi}^{\theta_0} i a \epsilon^{-1} \, d\theta = -\pi i \text{Res} \{f(z), z_0\}
\]

\( \mathcal{P} \oint_{C} f(z) \, dz = 2\pi i \sum_{n \text{ inside } C} \text{Res} \{f(z), z_n\} + \pi i \sum_{n \text{ along } C} \text{Res} \{f(z), z_n\} \)

If simple pole is along the corner of contour of 90°, then it only catch 1/4 of the residue.

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Residue theory to solve infinite sums
\( S = \sum_{n=-\infty}^{\infty} f(n) = \oint_{C} f(z) g(z) \, dz \) where \( g \) has simple poles at \( \mathbb{Z} \)

try \( g(z) = \frac{\pi \cot(\pi z)}{\sin(\pi z)} = \pi \cot(\pi z) \)

\( \text{Res} \{g(z), n\} = 1 \)

Contour: square: \( C_n : y = \pm \left( n + \frac{1}{2} \right), x = \pm \left( n + \frac{1}{2} \right), \lim_{n \to \infty} \oint_{C_n} f(z) g(z) \, dz \)

Contour integral \( = 0 \) if \( z f(z) \to \infty \) as \( |z| \to \infty \)

Then \( 0 = \sum_{n=-\infty}^{\text{poles of } f} \text{Res} \{f(z) g(z), z_k\} + \sum_{n=-\infty}^{\text{poles of } f} \text{Res} \{f(z) g(z); n\} \)

\( \therefore S = -\sum_{n=-\infty}^{\text{poles of } f} \text{Res} \{f(z) \pi \cot(\pi z), z_k\} \lim_{|z| \to \infty} z f(z) = 0 \)

\( S = \sum_{n=-\infty}^{\infty} \frac{1}{n^{2} + a^{2}} \quad a > 0 \)

\( S = -\text{Res} \left\{ \frac{1}{z^{2} + a^{2}} \pi \cot(\pi z); \pm ia \right\} = \frac{\pi}{a} \coth(\pi a) \)

Gaussian integrals
\( I_0 = \int_{-\infty}^{\infty} e^{-\frac{x^{2}}{a}} \, dx \)
\[ I_0^2 = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \, dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{(x^2+y^2)}{2}} \, dx \]

change to polars: \( \int_0^{2\pi} \int_0^{\infty} e^{-\frac{r^2}{2}} r \, dr \, d\theta = 2\pi \int_0^{\infty} e^{-\frac{r^2}{2}} \, r \, dr = 2\pi \int_0^{\infty} e^{-ru} \, du = 2\pi \left. \frac{e^{-ru}}{-u} \right|_0^\infty \)

if \( \text{Re}(a) > 0 \), then \( I_0^2 = \frac{2\pi}{a} \)

\[ I = \sqrt{\frac{2\pi}{a}} \]

\[ I_n = \int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} \, dx \quad \text{n is nonnegative integer} \]

\[ I_{n=\text{odd}} = 0 \]

integration by parts: \( dv = xe^{-\frac{x^2}{2}} \, dx, u = x^{n-1} \)

\[ = -\frac{e^{-\frac{x^2}{2}}}{a} x^{n-1} \bigg|_{-\infty}^{\infty} + \frac{n-1}{a} \int_{-\infty}^{\infty} x^{n-2} e^{-\frac{x^2}{2}} \, dx \]

\[ = \frac{n-1}{a} \frac{I_{n-2}}{\left(\frac{n-1}{2}\right)!} \]

\[ \sqrt{\frac{2\pi}{a}} \]

\[ n!! = n(n-2)(n-4) \ldots \]

second method: \( \frac{d}{da} \)

\[ (\ast) \text{ remember techniques: derivative, recursion.} \]

if \( a = -ib \)

\[ I_0 = \int_{-\infty}^{\infty} e^{\frac{ibx^2}{2}} \, dx = 2 \int_0^{\infty} e^{\frac{ibx^2}{2}} \, dx = 2 \left( \int_0^{\infty} \cos \left( \frac{bx^2}{2} \right) \, dx + i \int_0^{\infty} \sin \left( \frac{bx^2}{2} \right) \, dx \right) = 2 \sqrt{\frac{\pi}{b}} (C(\infty) + isgn S(\infty)) \]

\[ = \sqrt{\frac{2\pi}{|b|}} e^{i \text{sgn}(b) \frac{\pi}{4}} \]

\[ I_n = \frac{(n-1)!!}{(ib)^{\frac{n}{2}}} \sqrt{\frac{2\pi}{|b|}} e^{i \text{sgn}(b) \frac{\pi}{4}} \]

Fresnel integral

cornia spiral, S vs C plot.

Asymptotic estimation of integrals (stationary phase and saddle pt. methods.)

\[ I_k \int_{-\infty}^{\infty} A(x) e^{ik\phi(x)} \, dx \]

\( k \) "large"

envelope function \( A, k \) large \( \Rightarrow \) oscillates a ton except when \( \frac{d\phi}{dx} = 0 \). massive cancelation except near the "stationary points" \( x_j \), where \( \phi'(x_j) = 0 \)

for one stationary point, \( x = x_0 + \tau, \, dx = d\tau \)

\[ \int_{-\infty}^{\infty} A(x_0 + \tau) e^{ik(\phi(x_0) + \phi(\tau))} d\tau \]

expand \( A \) and \( \phi \) in taylor series.

\[ I_k = \int_{-\infty}^{\infty} \left[ A(x_0) + A'(x_0) \tau + A''(x_0) \frac{\tau^2}{2} + \ldots \right] e^{ik[\phi(x_0)] + \phi'(x_0) \tau + \phi''(x_0) \frac{\tau^2}{2} + \ldots} d\tau \]

It turns out that leading terms result from using truncated up to \( 0^{th} \) order of \( A \) and \( 2^{nd} \) order of \( \phi \).

\[ I_k = A(x_0)e^{ik\phi(x_0)} \int_{-\infty}^{\infty} e^{ik\phi''(x)} \frac{\tau^2}{2} d\tau \]

\[ = A(x_0)e^{ik\phi(x_0)} \sqrt{\frac{2\pi}{k|\phi''(x_0)|}} e^{i \text{sgn}(\phi''(x_0))} \]

if more than 1 stationary points, just do summation
$k$ large such that there are many oscillation b/w stationary points. 
method of stationary phase. 
$\phi(x_i) - \phi(x_j) \gg 2\pi$

how to get next order. 
expand $e^{ik\left(\phi''(x_0)\frac{x^3}{3!}\right)} = 1 + ik\left(\phi''(x_0)\frac{x^3}{3!}\right) - k^2\left(\phi''(x_0)\frac{x^3}{3!}\right)^2$

$= e^{ik(\phi(x_0))} \int_{-\infty}^\infty (\alpha + \alpha \tau + \alpha \tau^2 + \ldots) e^{ik\left(\phi''(x_0)\frac{x^2}{2}\right)}$

4 contribution of order $k^{-\frac{3}{2}}$

$I_k(p) = \int_{-\infty}^\infty e^{ik\left(\frac{x^3}{3!} - px\right)} dx, \quad P \in \mathbb{R}$
stat. pts. at $x = \pm \sqrt{p}$ only for $p > 0$
$\phi'' = 2x, \phi'''(x_{\pm}) = \pm 2\sqrt{p}$

$I_k(p) = \sqrt{\frac{2\pi}{k \cdot 2\sqrt{p}}} \left( e^{ik\left(\frac{e^2}{2} - p\frac{3}{2}\right) + i\pi \over 4} + e^{ik\left(\frac{e^2}{2} - p\frac{3}{2}\right) - i\pi \over 4} \right)$

$A_i(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\left(x^3 + \gamma \tau\right)} d\tau$
$\tau = kx^3, \tau = k^3 x, dx = k^2 d\tau$
$I_k = k^{-\frac{3}{2}} \int_{-\infty}^{\infty} e^{i\left(\frac{x^2}{2} - px\right)} d\tau = \frac{2\pi}{k^{\frac{3}{2}}} A_i\left(-k^{\frac{5}{2}}p\right)$

contribution of second type - edge effects
$I(k) = \int_a^b A(x) e^{ik\phi(x)} dx$
assumes no stationary points b/w a,b 
$e^{ik\phi(x)} = \frac{1}{ik\phi'(x)} \frac{de^{ik\phi(x)}}{dx}$
$I(k) = \frac{1}{ik} \int_a^b \frac{A}{\phi'(x)} dx e^{ik\phi(x)} dx$
integration by part, $dv = \frac{dx}{dx} e^{ik\phi(x)} dx$

$I(k) = \frac{1}{ik} \left[ \frac{A(x)}{\phi'(x)} e^{ik\phi(x)} \right]^b_a - \frac{1}{ik} \int_a^b \left( \frac{A}{\phi'} \right) e^{ik\phi} dx$
$I(k) = \frac{1}{ik} \left[ \frac{A(x)}{\phi'(x)} e^{ik\phi(x)} \right]^b_a + O(k^{-2})$

condition - no stationary point b/w a,b, or stationary points far enough away from edge.